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Outline

The Problem

The weighted Toeplitz least square (WTLS) problem

$$\min_{x\in\mathbb{R}^n} \|\Xi(Kx - f)\|_2^2 + u\|x\|_2^2$$

- $\Xi \in \mathbb{R}^{m\times m}$ is a weighting matrix (usually positive diagonal)
- $K \in \mathbb{R}^{m\times n}$ $(m \geq n)$ is a full-rank Toeplitz related matrix
- $\mu > 0$ is a regularization parameter
- f is a given right-hand side

☞ Applications lead to WTLS

- image reconstruction
- image restoration with colored noise
- nonlinear image restoration

Applications lead to WTLS

- Nonlinear image restoration

$$f = s(Kx) + \eta,$$

where f, x, and η represent the observed, the original image, and the noise vectors, respectively.

- $s(\nu)$ denotes a point nonlinearity
- K is a blurring matrix [NCT99]
  - BTTB : Dirichlet boundary condition
  - BCCB : periodic boundary condition
  - BTHTHB : Neumann boundary condition

[NCT99] M. K. Ng, R. H. Chan and W.-C. Tang, *A fast algorithm for deblurring models with Neumann boundary conditions*, SIAM J. Sci. Comput., 21 (1999), 851–866.

Equivalent Linear Systems – Normal Equations

$$\min_{x \in \mathbb{R}^n} \|\Xi(Kx - f)\|_2^2 + u\|x\|_2^2$$

- Normal equations of WTLS

$$(K^T \Xi^T \Xi K + \mu I)x = K^T \Xi^T \Xi f$$

  ☞ We can employ CG to solve this system

  ☞ Disadvantages:

  - condition number is squared
  - CG may converge slow
  - well-suited preconditioner based on fast algorithms is difficult to find (because of the weighting matrix)

Equivalent Linear Systems – Augmented System

- Augmented system associated with WTLS problem

$$\begin{bmatrix} W & K \\ K^T & -\mu I \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix},$$

where $W = (\Xi^T \Xi)^{-1}$ and $y = \Xi^T \Xi (f - Kx)$

☞ This is a generalized saddle point problem

☞ Many solution methods are available

- Uzawa, HSS, GSOR, ...
- preconditioned Krylov subspace methods

How to find a good preconditioner?

HSS Preconditioner [BN06]

Rewrite the augmented system into nonsymmetrix form

$$\begin{bmatrix} W & K \\ -K^T & \mu I \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \quad \text{or} \quad Au = c$$

- Hermitian and skew-Hermitian splitting [BGN03]

$$A = H + S$$

where

$$H = \begin{bmatrix} W & 0 \\ 0 & \mu I \end{bmatrix} \text{ and } S = \begin{bmatrix} 0 & K \\ -K^T & 0 \end{bmatrix}$$

[BN06] M. Benzi and M. K. Ng, *Preconditioned iterative methods for weighted toeplitz least squares problems*, SIAM J. Matrix Anal. Appl., 27 (2006), 1106–1124.

[BGN03] Z.-Z. Bai, G. H. Golub and M. K. Ng, *Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems*, SIAM J. Matrix Anal. Appl., 24 (2003), 603–626.

Examples: One-Dimensional Problem [BN06]

$K = [t_{ij}] \in \mathbb{R}^{n \times n}$ is a Toeplitz matrix defined by

$$\text{(i)} \quad t_{ij} = \frac{1}{\sqrt{|i-j|+1}} \qquad \rightarrow \textit{well} - \textit{conditioned}$$

$$\text{(ii)} \quad t_{ij} = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-|i-j|^2}{2\sigma^2}} \text{ with } \sigma = 2 \qquad \rightarrow \textit{ill} - \textit{conditioned}$$

Other parameters

- Ξ: positive diagonal random matrix with $\kappa_2(\Xi) \approx 10^3$
- $\mu = 0.001$
- stopping criterion: $\dfrac{\|c - Au\|_2}{\|c\|_2} < 10^{-7}$
- Initial guess: zero vector
- maximum iteration steps: 1000

Thanks!